Section - 13

1. P.T. Every integral Domain can be embedded into a field.

Proof: Let D be an integral domain and S be subset of $D \times D$ s.t.

$$S = \{ (a,b) \mid a,b \in D,\ b \neq 0 \}$$

Consider a relation $\sim$ in S s.t.

$$(a,b) \sim (c,d) \leftrightarrow ad = bc$$

Step I T.P $\sim$ is an Equivalence relation on S

i) Reflexive :- since D is Commutative ring

$\therefore ab = ba \quad \forall a,b \in D$

$\Rightarrow (a,b) \sim (a,b) \quad \forall (a,b) \in S$

$\sim$ is Reflexive.

ii) Symmetric :- Let $(a,b), (c,d) \in S$

$(a,b) \sim (c,d)$

$\Rightarrow ad = bc$

$\Rightarrow da = cb$ [$\because$ D is Commutative]

$\Rightarrow cb = da$

$\Rightarrow (c,d) \sim (a,b)$

$\therefore \sim$ is Symmetric.

iii) Transitive :- Let $(a,b), (c,d)$ & $(e,f) \in S$

$(a,b) \sim (c,d)$ & $(c,d) \sim (e,f)$

$\Rightarrow ad = bc$ & $cf = de$

$\Rightarrow adf = bcf$ & $bcf = bde$

$\Rightarrow adf = bde$

$\Rightarrow afd = bed$ $(\because D$ is Comm. ring$)$

$\Rightarrow af = be$

$\Rightarrow (a,b) \sim (e,f)$

$\therefore \sim$ is Transitive.

Hence $\sim$ is Equivalence relation on S.

Step II Consider field of Quotients

$$F = \left\{ \frac{a}{b} ; (a,b) \in S \right\}$$

T.P F is a field w.r.t. addition & mult.

Addition :- i) Addition is well defined :-

Let $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ elements of F

$$\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$$

$$\frac{(ad+bc)}{bd} = \frac{(a'd'+b'c')}{b'd'}$$

$$(ad+bc)\,b'd' = (a'd'+b'c')\,bd$$

$(ad+bc)b'd' = adb'd' + bcb'd'$

L.H.S

$= ab'dd' + bb'cd'$

$= ba'dd' + bb'dc'$

$= bda'd' + bdb'c'$

$= bd(a'd' + b'c')$ R.H.S.

$$(ad+bc)b'd' = bd(a'd' + b'c')$$

$\therefore$ Addition is well defined

ii) Associative :- let $\frac{a}{b}, \frac{c}{d}$ & $\frac{e}{f} \in F$

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \left(\frac{ad+bc}{bd}\right) + \frac{e}{f}$$

$$= \frac{adf + bcf + bde}{bdf} \quad \text{— (1)}$$

$$\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \left(\frac{cf+ed}{df}\right)$$

$$= \frac{adf + bcf + edb}{bdf} \quad \text{— (2)}$$

By (1) & (2)

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$$

$\therefore$ Associative Property holds.

iii) Existence of Identity :-

Let $\frac{0}{a}, \frac{c}{d} \in F$

$$\frac{0}{a} + \frac{c}{d} = \frac{0 \cdot d + ca}{ad} = \frac{ac}{ad} = \frac{c}{d}$$

$$\frac{0}{a} + \frac{c}{d} = \frac{c}{d}$$

$\therefore$ $\frac{0}{a}$ is Identity of $\frac{a}{b}$.

iv) Existence of inverse :-

Let $-\frac{a}{b}, \frac{a}{b} \in F$

$$\left(\frac{a}{b}\right) + \left(-\frac{a}{b}\right) = \frac{a \cancel{+} (-a)}{b} = \frac{a-a}{b} = \frac{0}{b} = 0$$

$$\therefore \left(\frac{a}{b}\right) + \left(-\frac{a}{b}\right) = 0$$

$\therefore$ $-\frac{a}{b}$ is Inverse of $\frac{a}{b}$.

v) Commutative :- Let $\frac{a}{b}$ & $\frac{c}{d} \in F$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$= \frac{da + cb}{db}$$

$$= \frac{cb + da}{db}$$

$$= \frac{cb}{db} + \frac{da}{db} \Rightarrow \frac{c}{d} + \frac{a}{b}$$

$\therefore$ Commutative property holds.

Step III Multiplication:-

i) Well defined :- Let $\frac{a}{b} = \frac{a'}{b'}$

& $\frac{c}{d} = \frac{c'}{d'}$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'}$$

$$\frac{ac}{bd} = \frac{a'c'}{b'd'}$$

$$(ac)b'd' = bd(a'c')$$

Now $(ac)b'd' = ab'cd'$

$= ba'dc'$

$= (bd)a'c'$

$\therefore$ multipl. is well defined.

ii) Associative :- Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \left(\frac{ac}{bd}\right) \cdot \frac{e}{f}$$

$$= \frac{(ac)e}{(bd)f}$$

$$= \frac{a(ce)}{b(df)}$$

$$= \frac{a}{b} \cdot \frac{(ce)}{(df)}$$

$$= \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$$ Associative Property holds.

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iii) Existence of Identity :- Let $\frac{1}{1}, \frac{a}{b} \in F$

$$\frac{1}{1} \cdot \frac{a}{b} = \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b}$$

$\therefore \frac{1}{1}$ is Identity of $\frac{a}{b}$.

iv) Existence of Inverse : Let $\frac{a}{b}, \frac{b}{a} \in F$

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = 1$$

$\therefore \frac{b}{a}$ is Inverse of $\frac{a}{b}$.

v) Commutative :- Let $\frac{a}{b}, \frac{c}{d} \in F$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{c \cdot a}{d \cdot b} = \frac{c}{d} \cdot \frac{a}{b}$$

Commut. Prop. holds.

<u>Step IV</u> Let us define a mapping

$$f : D \to F$$

$$f(a) = \frac{ax}{x} \quad \forall\, a \in D$$

<u>T.P</u> f is an isomorphism.

i) Well defined :- Let $a, b \in D$

$$a = b$$

$$\frac{ax}{x} = \frac{bx}{x}$$

$f(a) = f(b)$ $\therefore f$ is well defined.

ii) One - One :- Let $a, b \in D$

$$f(a) = f(b)$$

$$\frac{ax}{x} = \frac{bx}{x}$$

$$ax^2 = bx^2$$

$$(a-b)x^2 = 0$$

$$a-b = 0$$

$$a = b$$

$\therefore f$ is one-one.

iii) Onto :- Let $\frac{ax}{x} \in F$ $\exists$ $a \in D$

$$f(a) = \frac{ax}{x}$$

$\therefore f$ is Onto.

iv) homomorphism :- Let $a, b \in D$

$$f(a+b) = \frac{(a+b)x}{x}$$

$$= \frac{ax+bx}{x}$$

$$= \frac{ax}{x} + \frac{bx}{x} \Rightarrow f(a)+f(b)$$

$$f(ab) = \frac{(ab)x}{x} \Rightarrow \left(\frac{ax}{x}\right)\left(\frac{bx}{x}\right)$$

$$= f(a)f(b)$$

$\therefore f$ is homomorphism.

$\therefore f$ is isomorphism.

Hence An integral Domain Can be embedded into a field.